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J. Phys. A: Math. Gen., Vol. 12, No. 6, 1979. Printed in Great Britain

The number of infinite red-shift surfaces of the generalised Kerr–Tomimatsu–Sato metric

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Received 8 August 1978, in final form 2 November 1978

Abstract. A proof that a certain polynomial has n real zeros is presented. From this it then follows that the generalised Kerr–Tomimatsu–Sato metric has $2n$ distinct infinite red-shift surfaces.

1. Introduction

The number of real roots of the determinantal polynomial (Yamazaki 1977, Dale 1978)

$$A_n = |a_{ij}|_n$$

where

$$a_{ij} = \frac{z^{i+j-1} + (-1)^{i+j}c}{i+j-1}$$

and c is an arbitrary negative constant is of significance for the Kerr (1963) and Tomimatsu–Sato (1973) class of metrics. Since in this case the norm of the Killing vector (∂t) is A_n/B, an infinite red-shift surface will occur when $A_n = 0$. In this paper it will be shown that A_n has n distinct real positive zeros, from which it then follows that there are $2n$ such surfaces (Cosgrove 1977).

In the theorems which follow, use will be made of Jacobi's theorem in the following form.

Let

$$D_n = |d_{ij}|_n$$

be a symmetrical determinant of order n ($\geqslant 2$); then

$$D_{11}D_{nn} - D_{1n,1n}D_n = D_{1n}^2 \tag{1}$$

where D_{ij} is the cofactor of d_{ij} and $D_{1n,1n}$ denotes the determinant obtained from D by deleting the first and nth rows and columns.

2. The number of real zeros of A_n

Lemma. If

$$P_n = |\alpha_{ij}|_n$$

where

$$\alpha_{ij} = \frac{z^{i+j} - c(-1)^{i+j}}{i+j}$$

then $P_n > 0$.

Proof. Let

$$\alpha = (\alpha_{ij})_n \qquad \beta = (\beta_{ij})_n,$$

where

$$\beta_{ij} = \binom{i-1}{j-1}\xi^{i-j} \qquad (i \geqslant j) \qquad \text{and} \qquad \beta_{ij} = 0 \qquad (i < j),$$

and form the matrix product $\beta\alpha\beta^{\mathrm{T}}$ where the T denotes the transpose. Let

$$\gamma = (\gamma_{ij})_n = \beta\alpha\beta^{\mathrm{T}}. \tag{2}$$

Taking determinants of (2) we get

$$P_n = P_n(z, 0, c) = P_n(z, \xi, c) \tag{3}$$

where

$$P_n(z, \xi, c) = |\gamma_{ij}|_n. \tag{4}$$

Differentiating the identity (3) partially with respect to z and then putting $\xi = -z$ we get

$$\dot{P}_n = zQ_{11} \tag{5}$$

where Q_{11} is the cofactor of λ_{11} in

$$Q_n = |\lambda_{ij}|_n$$

$$\lambda_{ij} = \frac{(1-c)z^{i+j} + c(z-i-j+1)(1+z)^{i+j-1}}{(i+j)(i+j-1)}$$

and the dot denotes differentiation with respect to z.

By applying Jacobi's theorem (equation (1)) to the determinant Q_n and using equations (3) and (5) we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{P_n}{P_{n-1}}\right) = z\left(\frac{Q_{1n}}{P_{n-1}}\right)^2 \tag{6}$$

whenever $P_{n-1} \neq 0$.

Now, by assuming that $P_{n-1} > 0$ and noting that when $z = 0$

$$P_n/P_{n-1} > 0$$

it follows from (6) that $P_n > 0$. But $P_1 = \frac{1}{2}(z^2 - c)$ and hence the lemma follows by induction.

Theorem. The determinantal polynomial A_n has exactly n distinct real positive zeros.

Proof. Applying Jacobi's theorem to the determinant A_n we get

$$A_{11}A_{n-1} - A_{1n,1n}A_n = P_{n-1}^2 > 0 \qquad \text{(lemma)} \tag{7}$$

and from (7) it follows that A_n and A_{n-1} cannot have a common real zero.

Now, if

$$B_n = |b_{ij}|_n$$

where

$$b_{ij} = \frac{c(1+z)^{i+j-1} + (1-c)z^{i+j-1}}{i+j-1}$$

then

$$A_n = B_n \tag{8}$$

$$\dot{A}_n = B_{11} \qquad \text{(the cofactor of } b_{11}\text{)} \tag{9}$$

(see Dale 1978). By applying Jacobi's theorem to the determinant B_n and using equations (8) and (9) we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{A_n}{A_{n-1}}\right) \geqslant 0 \tag{10}$$

whenever $A_{n-1} \neq 0$. Also, when $z = 0$

$$A_n/A_{n-1} < 0 \tag{11}$$

and for large z

$$A_n/A_{n-1} \sim k^2 z^{2n-1}. \tag{12}$$

If we now assume that the real zeros of A_{n-1} occur at $z = z_i$, $i = 0, 1, \ldots, (n-2)$, and that $0 < z_0 < z_1 \ldots < z_{n-2}$, then since A_n and A_{n-1} have no common real zero it follows from equations (10)–(12) that in each of the intervals $0 < z < z_0$, $z_i < z < z_{i+1}$, $(i = 0, 1, \ldots, n-3)$, $z > z_{n-2}$, A_n has one and only one real zero and none in the interval $z < 0$. Hence, if the theorem is true for A_{n-1} it is true for A_n. But $A_1 = z + c$, which has one real positive zero. Therefore, by induction, A_n must have n distinct real positive zeros. Further, from equation (7) we see that A_{11} and A_n cannot have a common real zero. But since (Dale 1978)

$$(1+z)\dot{A}_n - n^2 A_n = (1-c)A_{11}$$

it follows that A_n and $\dot{A}_n$ cannot have a common real zero, i.e. none of the real zeros of A_n is repeated.

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